Early Algebraic Language and the Algorithm: Preparing an Art of Thinking for People and Machines

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In the history of the algorithm we begin with the term, *Algorismus*, which in the Middle Ages (starting in the 13th century) meant calculating with the four operations on integers by using Arabic numbers on paper, as the great 9th century Baghdad algebraist Al-Kwarizmi had explained in his work on arithmetic. This meant to mobilize a series of graphic and cognitive techniques for the four operations, an abacus in writing.

In the fifteenth-century, with the printing press, mathematicians expressed and spread in writing some new practices of calculation. This new forms of calculation have promoted codification and standardization and involved some new cognitive strategies. The arrival point of this process has been the technique of forming an equation for the solution of a class of problems. This technique led to extend the uses of this codification in many different disciplines and in this way to unify a diverse collection of mathematical arts, from perspective to mechanics, as Luca Pacioli wrote already in 1494. Such unification was at the level of practice as well as at the level of theory and symbols. We can see a continuous line from sixteenth-century algebraic notation, language and structures to Boolean algebras and binary numeration to build algorithms. Born as an art of thinking, a method to lead one's mind correctly, this *mathesis universalis* or algebraic language of sciences gave life to the motivation of giving a solution of any sort of problem. In order to do so, the procedures of solutions should be as uniform and as general as possible, what happened in calculating machine. From Tartaglia, Descartes, Pascal and Leibniz, the project became then to automatize this same thinking, as in Boole. We can say that today computers and artificial intelligence realize these early modern projects of automatizing calculation by introducing a shift: from the rules for the direction of mind and deterministic logic to AI, probabilistic logic and machine learning.

The previous workshops of this project on the history of algorithm in modernity have discussed the theses above. To open a discussion on the 22nd and given that I am an historian of early modern mathematics at the beginning of this new project, I distribute a paper concerning early modern algebra as a sort of preliminary reading: it contains some points useful for our purpose. I shall explain in my talk their connections to history of algorithms. I take Michael Mahoney's work as a point of departure not so much because he has been my Doktorvater but because I believe in this project it is important to see historiography as connected to contemporary research in computer science, and he wrote this text in the late sixties. It is noticeable that after writing on Fermat, in the eighties he moved to history of computing.

The Path towards the Royal Road:
Algebra and Method in the Sixteeenth-century

(revised version of a lecture)

The Royal Road is the title Michael Mahoney's PhD dissertation, presented at Princeton more than fifty years ago, in May 1967. Never published, it traces the mathematical, logical and philosophical motives leading to Viète's symbolic algebra. Let us follow this text, its contribution and the new facts and interpretation we have acquired on this topic in the last fifty year.

The very expression of the Royal Road comes from Proclos (412-482). This famous *topos* concerns a road, the ancient royal road connecting in early antiquity Smyrne to Persepolis, and a person, Ptolemy I Soter, Alexander's general who supported Euclid's endeavour. According to a famous anecdote mentioned by Proclos (412 – 485) and repeated by Petrus Ramus (1515-1572), Ptolemy the general once asked Euclid whether there was any shorter way to the knowledge of geometry than by the study of the *Elements*. Euclid answered that there was no royal road to geometry. In attempting to trace the sources of sixteenth-century symbolic algebra Mahoney looked at medieval algebra within a humanistic perspective: early modern mathematicians considered the Arabic mathematical art a heuristic means in mathematics that the Greek could not have ignored, the shortcut in geometrical invention that they had kept secret.

In the first part of *The Royal Road* we see the parts of Greek mathematics related to algebraic contents and methods: geometrical analysis and geometric algebra of the Greeks. For, algebra differs from the most known pages of Greek mathematics, but is related to other aspects of it, in two major ways. First, because algebra uses analytical reasoning, the logical process in which we start by taking the result as granted and the argument develops its consequences: this was quite common in problems but less in theorems or demonstrations. Second, because algebra calculates with the four operations on quantities indeterminate quantities. Algebraic quantities in medieval algebra are indeterminate in two different ways: because at least one is unknown and needs to be found, and because it deals with different species of things in the same problem. Things expressed in an equation, as in the rule of three, can be for instance arithmetical and geometrical (in problems or geometrical measures), or can be commodities and money (in bartering problems), months and money (in problems of simple and compound interest), pound and pence (in problems dealing with metrological transformations).

Greek geometry included of course arguments on indeterminate quantities. According to Euclid's *Elements* mathematicians can perform calculations, that is the four operations mostly in arithmetic, as well as in the theory of ratios and proportions. In the second Book however we find a calculation on segments which is one sort of calculation on indeterminate quantities. This book contained a summary of the application of areas, a theory of Pythagorean origin. Many historians have noticed the special features of this book, such as M. Cantor, P. Tannery, H. G. Zeuthen and T. Heath. Progressively, some of these authors and a generation of historians very informed of recent algebraic discoveries introduced the idea that the Greeks had algebra but they did not accept it for philosophical reasons. They called it geometric algebra, a historiographical idea very active among historians of mathematics for more about a century. Also, according to the historian of Babylonian algebra Neugebauer, a mathematician who had been studying mathematics in Gottingen with Emmy Noether, the Greeks actually took algebra from the Babylonians, while Van der Waerden went further and accepted the historiographical tradition of the Greeks (Proclos and Eudemos) saying that in fact the Pythagoreans themselves were the transmitters through application of areas. The story was that after the discovery of irrational numbers Pythagoreans developed not only the theory of ratios and proportions, but also application of areas. For both Neugebauer and Van der Waerden, application of areas was algebra in geometrical cloths.[1]

A long-standing twentieth-century historiographical debate has been about whether an algebraic doctrine was in fact hidden in the theory of the application of areas or if this theory was genuinely geometrical, even though useful for calculation, and algebra a late consequence of it. A first phase of this debate culminated with the essays by Jacob Klein, a philosopher well-versed in mathematics and logic (*Greek mathematical thought and the origin of algebra*, published in German in 1934 and 1936, while Turing developed his machine). While it got published in Neugebauer's journal, these essays contain a philosophical analysis of algebraic thinking that stressed the *rupture épistémologique*, the radical discontinuity between Greek and modern algebra. Klein went beyond the early modern nostalgia of golden age Greece and revealed that early modern people had a radically different way of conceiving number and indeterminate quantities with respect to their classical heroes. Mahoney agreed on this crucial point and he contributed to this debate about analysis and geometric algebra of the Greeks by making two substantial steps. First, he showed that Greek mathematical analysis should not so much be seen as a type of mathematics following a different logic, but rather as a mathematical corpus playing the role of a "toolbox" for problem-solving in mathematics. Second, he provided evidence that Renaissance mathematicians were not only aware of this role of the analytical corpus but also that they conceived, virtually for the first time in history, the existence of Greek "geometric algebra" consisting mostly in the application of areas theory presented in Book II of the *Elements*. In the sixteenth-century, this theory was considered as a lost body of knowledge making explicit the art of discovery, and hidden for this reason. Already In the course of the Middle Ages some arithmetical and algebraic authors had taken inspiration from Book II of the *Elements* to articulate their argument on numbers or justify it geometrically. In other words, this computation on segments, purely geometrically conceived, became associated with arithmetic, in particular, the first ten propositions,

[1] A recent reconstruction of the whole history of this debate is in Jens Hoyrup 2017, https://www.aimspress.com/fileOt-02-00128.pdf

corresponding to the remarkable products. By the time Campanus of Novara wrote his translation, the Euclidean arithmetical books was transformed by this interpretation of Book II. Other translation kept the original geometrical sense, but Campanus' version was the best known, and especially present in earlier printed editions.[2] Later, some new philological finding such as Proclos' *Commentary* justified this belief in the sixteenth-century, because a few ancient texts mentioned a general method for solving mathematical problems.

I found more examples of this belief in sixteenth-century mathematicians. It was widespread because it was a way to justify a practical, basic mathematical art, a sort generalisation of the rule of three, as a full mathematical discipline, even an art of thinking susceptible to be applied to any sort of problems.

Both points converged in the main thesis Mahoney formulated about the process leading to the constitution of symbolic algebra: what made the invention of symbolic algebra possible was the combination of three elements. First, Greek geometrical analysis -the toolbox- and its algebraic interpretation, together with Book II; second, the rediscovery of another Alexandrian mathematician, Diophantos (fl. 200 A.D.), who had developed a problem-solving oriented kind of arithmetic; third, cossist algebra, the arithmetical problem solving technique of Arabic origin transmitted by Renaissance Italian mathematicians. But how were the three elements combined? In Mahoney's view, **a particular sixteenth-century man** unified all of these components creating the conditions of the invention of an algebra of general quantities. This author is famous even though mostly remembered as a teacher of mathematics and not as a creative mathematician, that is Petrus Ramus.

Mahoney thought that the role of Ramus on sixteenth-century intellectual history had been underestimated. He wrote:

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[2] Leo Corry "Geometry and Arithmetic in the Medieval Tradition of Euclid's Elements: a view from Book II. " *Archive for History of Exact Sciences* Vol. 67, No. 6 (November 2013), pp. 637-705

The Ramist element in European thought has been largely ignored by intellectual historians, despite the work of Perry Miller and W. S. Howell, who have revealed its strength in New and old England, respectively, and of Father Ong…One reason, perhaps, that Ramism has received short shrift from intellectual historians is that it is often hard to identify separately. As Father Ong takes pains to point out, Ramism was not so much a philosophy as a set of attitudes towards philosophizing. Most of these attitudes are reflected in specific doctrines which, though not all original with Ramus, nonetheless take on special meaning as part of the Ramist complex of ideas.

As to the actual role of Ramus's thought, and concerning the places and institutions where he was influential, Mahoney adopts the term of "undercurrent":

There are at least two approaches to the intellectual history of sixteenth-century Europe. The one studies the works of the great minds of the century, the religious writings of Luther…and others. But the great Christian controversies of the age, or even the scepticism of a Montaigne, tend, I think, to hides the subtler and, in the final analysis, more influential undercurrents of the period, undercurrents which would surface in the seventeenth-century and sweep European thought in their direction. The second approach aims at elucidating these undercurrents. Two of them may be characterized as the activity of the workshop and the activity of the classroom. (p.157)

And p. 159:

Colleges shaped European thought in ways of which the intellectual historian is only now becoming conscious.

Yet, Ramus's mathematical competence had been questioned by historians. Among his contemporaries Ramus was famous for his theories on the rhetorical reform of logic and for his textbooks, including mathematical textbooks. However, he did not hide his limits in the topic. He wrote:

I confess that nothing has ever been written by the human hand more obscure than the fifteen books of Euclid's Elements. I confess moreover that, when I came to Euclid, the study of the previous arts seemed to me not study but mere sport. (

Mahoney took the risk of believing in Ramus' competence:

Ramus was not speaking from ignorance of mathematics when he wrote this passage. If he had found Euclid difficult, he had not been dissuaded from further study. By 1569, the year in which he composed the *Scholae mathematicae*, he was probably the mathematically most literate man in Europe. He had read, either in Latin translation or Greek manuscripts, all of the Greek mathematicians and their commentators. He had volumes to speak on the subject, and one must, I think, suspend judgement on his abilities (as Ong has not done) until those volumes have been carefully studied.

Ramus's had a lot to say about mathematics. In Mahoney's account,

According to Ramus Greek mathematicians such as Euclid excelled in mathematics but failed as teachers. To him, the Royal Road meant the path of effective pedagogy, and he intended to lay out that path. Euclid failed because he did not include the analytical tradition, and what he included was often redundant.

Mahoney points to what Ramus considered effective pedagogy, by interpreting Ramus's notions of invention as analysis:

Method was the process by which the results of analysis were laid before the students in such a way that they could grasp the interrelationships of those results, interrelationships which themselves had been simplified. Art was the technique that the students learned in order to emulate the material they had been taught. The model for such activity was grammar and rhetoric. Analysis took apart a Ciceronian oration. Method showed how it was put together. What the students learned was the art of rhetoric. RR p.171

Mahoney's theses are now part of the mainstream view of the mathematical transformations of the sixteenth-century. What I would like to stress here are the new facts and interpretations which modify but also confirm Mahoney's main theses, as well as the meaning of his standpoint in the context of twenty-century historiography.

My understanding of the story

When I started working on this sixteenth-century genesis of Western early modern algebraic thinking I chose to look at rhetorical strategies of understanding (how to make a problem susceptible of solution, how to give it the right shape for that). By rhetorical strategies I do not refer to rhetoric of persuasion but rhetoric of cognition, as it was practiced by sixteenth-century authors, teachers and students.

Starting on this ground, in my research I have been able to establish a few new facts.

1) François Viète was unique in his genius but not in his topic. His symbolic algebra was new but had some precedents. A small group of French algebraist are responsible for the constitution of a French algebraic tradition acknowledged as such in the rest of Europe, that is, in particular, Jacques Peletier (1517 – 1582) and Guillaume Gosselin (fl. 1577- 1583). In fact Viète was Gosselin's patron. This French algebraic tradition was substantial and important for further developments.
2) The rhetorical reform of logic in the sixteenth-century is crucial in the constitution of symbolic algebra. The transformation of *cossist* algebra into a "short and clear science", in "*subtilior arithmetica*" and finally in analytic art in the sixteenth-century can be understood if we follow the changes introduced by early French algebra books: order, notation, role of problems, structure of solutions, *analysis* as rhetorical structure. The most important change was to accept that mathematics could follow a "rhetorical" logic, that is a logic having various degrees of probability, like in Cicero. In other words, mathematical certainty did not depend on a more cogent structure, but on the nature of mathematical objects. The logic of this like of any other science is that of the so called abbreviated syllogisms i.e. the syllogism in which only one premise is explicit. This kind of syllogism was also called dialectical, rhetorical syllogism or enthymeme already in Aristotle and in Stoic logic.
3) I confirmed the importance of Ramus in this story. First of all, I found in Ramus' *Scholae Mathematicae* (1569) an explicit mention of geometric arithmetic as being the object of Book II and Book II being logically independent and a good candidate for being, logically, Book I. Ramist dialectic was crucial in the very concepts of early modern algebra. The mathematicians following Ramus' reform introduced

Invention and Disposition (the first and second parts of oration) in mathematical books, so that they shaped algebra as a classical discipline and not as a so that it could not be call *cossist* anymore. *Problema, quaestio, aequatio*, etc. were consciously used with a dialectical meaning as well as with the mathematical one. The logic emerging from the rhetorical reform of logic was quite present in the minds of French algebraists, to the point that it actually motivated them to raise this minor discipline to the role of a *mathesis* directly based on common notions. As Mahoney had put it thinking of the time after Viète " Ramus imposed on the discipline philosophico-pedagogical demands which, as it turned out, algebra was best suited to meet. Or perhaps one should say that Ramus, or Ramism, exercised immense influence on a certain group of mathematicians who, led by François Viète, in turn shaped algebra and mathematics to meet those demands. (RR p. 148)"

4) But when looking at things in this way, Ramist dialectic appears as a catch phrase subsuming the larger phenomenon of the rhetorical reform of logic. It is this larger and greatly influential phenomenon, not a single individual, which can be considered responsible for the unification of the components necessary for symbolic algebra. Jacques Peletier is a good example of an author which cannot be defined a Ramist but who developed, often before Ramus, innovations in the theory and the practice of mathematics which later converged with Ramism. He was an accomplished mathematician as well as a humanist engaged in writing sciences in the vernacular. For Peletier, the foundation of an art, and most particularly of mathematical arts, are natural light, invention and disposition. He was the first to write that algebra is an art of thinking and speaking ("*elle apprend à discourir*"). Besides, Peletier thought that looking for syllogisms in Euclidean demonstrations was most of all silly. Proofs can be syllogisms or enthymemes, He did write some proofs in his version of Euclid, but explaining the connections between statements more than proving their soundness. Often, Ramus did not provide proofs also because he considered more important that students find them.

5) Another algebraist, Guillaume Gosselin, in 1577 built his algebra on common notions and enthymemes. This is a first version of Viète's set of axioms and demonstrations. This clarifies, it seems to me, what we should understand for the use, by Viète, of Ramus's three laws. These defined Ramus' method and derived from the *Posterior Analytics*: Every theorem should be universal and necessary. Every theorem belongs to one and only theory. Any specific theorem should follow (and not precede) a more general one. The important point is not so much to what extent the three laws are effective, but that they are the only requirement, the only actual logical structure of Ramus' dialectic. The emphasis was not on demonstrating truth, but on reaching truth by the faculties of the mind. The path to truth in what they called *mathesis* included three aspects: natural light, i.e. faculty of counting, the Ciceronian common notions (the axioms of the stoic logic, identified with Euclid's common notions or axioms) and finally the abbreviated syllogism. Gosselin provided what he called an arithmetic proof of the solution formula for equations.

6) Proclos is a main character in this story because his *Commentary to the First book of Euclid's Elements* was a reference in the sixteenth-century. Proclos does not characterize a side of sixteenth-century mathematical debates. In fact, all kinds of mathematicians, for instance Ramus in spite of his criticisms, but also Ramus's opponents, are influenced by Proclos views: on *mathesis*, on common notions, on the definition of proposition, on the relation between problems and theorems, on the role of problem solving sort of mathematics. My forthcoming book is about debates among different social categories of mathematicians: teachers in the *Artes*, court cosmographers and university professors. They disagree especially on the role of algebra and of abbreviated syllogisms. However, their common ground is a Proclean view of mathematics and of the mathematical disciplines. The origin of this shared view is the Euclid's *editio princeps* by Grynaeus published in Basel in 1533, in which Proclos' commentary on Euclid was also published. Proclos was also an authority for reading of Euclid combining the arithmetical and the geometrical

books and looking for a *mathesis* prior to arithmetic and geometry. This is a confirmation of Mahoney's thesis according to which the idea of geometric algebra was a sixteenth-century idea, but it makes clear that this idea cannot be identified with Ramus, but with a larger movement, from Grynaeus and Melanchthon to Kepler.

7) This explains, it seems to me, what we should understand for the use of Ramus' three laws. The important point is that they are the only requirement. For the new logicians, this was the path to truth to be recommended because it reflected directly the natural process of understanding. It is the process of knowing according to the laws of thought; it is as natural as the grammar of natural languages for native speakers.
8) A proper argumentation, in this setting, is constituted of a *quaestio* and a common notion. First we should transform the *quaestio* and give it the proper form, then we should apply a common notion to that question. This is what we can also find in Descartes's *Regulae.*

Historiography

In history of mathematics, Mahoney's dissertation made its appearance against the background of the debate concerning Greek geometric algebra. But the question about analysis needs to be put in the context of twentieth-century mathematics. By the late sixties, algebraic structures were the way to understand all parts of mathematics. But algebra was one of these parts and analysis was another, the two did not have much in common. For instance, algebraic topology suggested that a sheave is an effective context for analysis, but its logic will be in general *intuitionistic* and not classical. Mahoney clarified the way in which analysis meant an entirely different discipline in the early modern times. He also gave very important contribution to the history of algebra and of *algebraic thought.* He even characterized algebraic thought as it can be traced back to the Babylonian. Yet, he was looking for a continuity we now are more ready to question.

The stake was whether to give priority to the progress of algebra through centuries or to interpret a particular mathematician, such as Viète and Fermat, in the light of the intellectual and social life of his time.

Obviously, this debate was mostly a professional contrast opposing history of mathematics for mathematics versus history of mathematics for its own sake. This same opposition lead to the destructive criticism by André Weil.

At this point, stakes have changed.

First, we can focus on authors who appear at first less flamboyant mathematicians, and we can look at famous mathematicians insofar as they share practices with minor figures.

Second, we can acknowledge fully the role of Proclos in the sixteenth-century, even though this influence is more at the level of a common language than a direct source of mathematical or philosophical achievement. Among the greatest mathematician and an astronomer in the fifth-century, Proclos was not seen as such by historians of mathematics. Proclos mentioned the whole story in his *Commentary to the first book of Euclid's 'Elements'* to stress another way of doing mathematics, more apt to develop human intellectual apprehension of truth. This other way of doing mathematics was freed of what Ramus called the *insanum demonstrandi stadium* that is, the insane zeal for demonstration typical of classical Greek geometry, but opened the Pandora's box of a new *insanum calculandi stadium*.